Heisenberg's wave packet reconsidered

J. Orlin Grabbe*

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This note shows that Heisenberg's choice for a wave function in his original paper on the uncertainty principle is simply a renormalized characteristic function of a stable distribution with certain restrictions on the parameters. Relaxing Heisenberg's restrictions leads to a more general formulation of the uncertainty principle. This reformulation shows quantum uncertainty can exist at a macroscopic level. These modifications also give rise to a new form of Schrödinger's wave equation as the equation of a vibrating string. Although a heat equation version can also be given, the latter shows the traditional formulation of Schrödinger's equation involves a hidden Cauchy amplitude assumption.

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A generalized wave packet

We begin by showing that Heisenberg's choice for a wave function in his original paper [4] on the uncertainty principle is simply a renormalized characteristic function of a stable distribution, $S_{\alpha,\beta}(x; m, c)$ with $\alpha = 2$ and $\beta = 0$, and location and scale parameters m and c. Relaxing the assumptions on α, β so that $0 < \alpha \leq 2, \beta \neq 0$, leads to a more general formulation of the uncertainty principle. These modifications also give rise to a new form of Schrödinger's partial differential equation.

Consider the following wave packet $\psi(x,t)$, where at time $t = 0$, $\psi(x,0)$ has the form

$$\psi(x,0) = A_o \exp[imx - c|x|^\alpha], \tag{1}$$

where

$$A_o = [\frac{\alpha(2c)^{\frac{1}{\alpha}}}{2\Gamma(\frac{1}{\alpha})}]^{\frac{1}{2}}. \tag{2}$$

It is easy to see that $\psi(x,0)$ is normalized to unity:

$$\int_{-\infty}^{\infty} \psi * (x,0)\psi(x,0)dx = A_o^2 \int_{-\infty}^{\infty} \exp[-2c|x|^\alpha]dx = 2A_o^2 \int_0^{\infty} \exp[-2cx^\alpha]dx. \tag{3}$$

Using the relation

$$\int_0^\infty y^k e^{-y^\alpha} dy = \frac{1}{\alpha}\Gamma(\frac{k+1}{\alpha}) \tag{4}$$

and making the substitution $u = (2c)^{\frac{1}{\alpha}}x$, we obtain

$$2A_o^2 \int_0^\infty \exp[-2cx^\alpha]dx = \frac{2A_o^2}{(2c)^{\frac{1}{\alpha}}} \int_0^\infty e^{-u^\alpha} du = \frac{2A_o^2}{(2c)^{\frac{1}{\alpha}}} \frac{1}{\alpha}\Gamma(\frac{1}{\alpha}) = 1. \tag{5}$$

Now, the form of the wave packet in Eq.(1) can be compared to Heisenberg's original wave packet, denoted here $H(x, 0)$:

$$H(x, 0) = (2\tau)^{\frac{1}{4}} \exp[2\pi i\sigma_o x - \pi\tau x^2]. \tag{6}$$

If we make the substitutions

$$2\pi\sigma_o = m \tag{7}$$

$$\pi\tau = c \tag{8}$$

$$\alpha = 2 \tag{9}$$

in $\psi(x, 0)$, the wave packet of Eq.(1), we obtain $H(x, 0)$. (Note that with $\alpha = 2$, $A_o = [\frac{2(2c)^{\frac{1}{2}}}{2\Gamma(\frac{1}{2})}]^{\frac{1}{2}} = (\frac{2c}{\pi})^{\frac{1}{4}} = (2\tau)^{\frac{1}{4}}$.)

Now let's derive the amplitude function of $\psi(x, 0)$, which will necessarily also give us the amplitude function of $H(x, 0)$. First note that the log characteristic function of a stable distribution is

$$\log\,\varphi(z) = \log \int_{-\infty}^\infty \exp[ixz]dF(\frac{x-m}{c'}) \tag{10}$$

$$= imz - |c'|^\alpha|z|^\alpha[1 + i\beta(z/|z|)\tan(\pi\alpha/2)],\ \text{if } \alpha \neq 1 \tag{11}$$

$$= imz - |c'|^\alpha|z|^\alpha[1 + i\beta(z/|z|)(2/\pi)\log|z|],\ \text{if } \alpha = 1 \tag{12}$$

where m is a real number, $c' \geq 0$, $0 < \alpha \leq 2$, $|\beta| \leq 1$. Proof of this theorem, due to Khintchine and Lévy in 1936, may be found in [5] or [3]. Here α, the *characteristic exponent*, is essentially an index of peakedneess ($\alpha = 2$ for the normal or Gaussian distribution, $\alpha = 1$ for the Cauchy distribution). The parameter β is an index of skewedness ($\beta = 0$ for symmetric distributions). The parameter $c' = c^{\frac{1}{\alpha}}$ is a scale parameter (the standard deviation when $\alpha = 2$). Finally, m is a location parameter (the mean if $\alpha > 1$; it is also the median or modal value of the distribution if $\beta = 0$).

For $\beta = 0$ we obtain the characteristic function of a *symmetric* stable distribution, which is identical to Eq.(1), if we omit the normalizing constant A_o. Therefore, for the amplitude function of our wave packet, we take the Fourier transform, $A(z)$, of Eq.(1) to obtain

$$A(z) = \int_{-\infty}^{\infty} \psi(x,0)\, \exp[-ixz]dx = A_o \int_{-\infty}^{\infty} \exp[imx - c|x|^{\alpha}]\, \exp[-ixz]dx = A_o s_{\alpha,0}(z;m,c). \tag{13}$$

In other words, we obtain a symmetric stable density function $s_{\alpha,0}(z;m,c) = dS_{\alpha,0}(z;m,c)$ with the normalization constant A_o for the amplitude function. The symmetric stable density has $0 < \alpha \leq 2$, $\beta = 0$, and location and scale paramters m and c, respectively. Note that the amplitude function is normalized so that the integral of its *square* is equal to 1. This involves the square of the probability density function $s_{\alpha,0}(z;m,c)$.

For Heisenberg's case where $\alpha = 2$, we may explicity solve for $A(z) = A(\sigma)$, which will be necessarily the Gaussian density multiplied by a normalizing constant. We reintroduce a factor of 2π to obtain

$$A(\sigma) = \int_{-\infty}^{\infty} (2\tau)^{\frac{1}{4}}\, \exp[2\pi i\sigma_o x - \pi\tau x^2]\, \exp[-2\pi ix\sigma]dx = (\frac{2}{\tau})^{\frac{1}{4}}\, \exp[-\frac{\pi(\sigma-\sigma_o)^2}{\tau}]. \tag{14}$$

This is Heisenberg's amplitude function. That the integral of its square is 1 follows from:

$$\int_{-\infty}^{\infty} [A(\sigma)]^2 d\sigma = (\frac{2}{\tau})^{\frac{1}{2}} \int_{-\infty}^{\infty} \exp[-\frac{2\pi(\sigma-\sigma_o)^2}{\tau}]d\sigma = (\frac{2}{\tau})^{\frac{1}{2}}(\frac{\tau}{2\pi})^{\frac{1}{2}} 2\int_{0}^{\infty} e^{-u^2}du = \frac{\sqrt{\pi}}{\sqrt{\pi}} = 1, \tag{15}$$

where we have used the substitution $u = \sqrt{\frac{2\pi}{\tau}(\sigma - \sigma_o)}$. Note that the usual normalizing constant $\frac{1}{d\sqrt{2\pi}}$ for the Gaussian distribution (where d is the standard deviation) has been absorbed into A_o. So above and below, when we write the stable density $s_{\alpha,\beta}(z;m,c)$, we will understand the omission of the usual normalizing constant, and will consider only the normalizing A_o in the product $A_o s_{\alpha,\beta}(z;m,c)$. This will ensure that the square of the amplitude function is a probability distribution.

Alternative amplitude functions

For $\alpha = 1$, which corresponds to the Cauchy distribution, the normalizing constant $A_o = c^{\frac{1}{2}} = (c')^{\frac{\alpha}{2}} = (c')^{\frac{1}{2}}$, so the amplitude function is

$$A(z) = A_o s_{1,0}(z;m,c) = c^{\frac{1}{2}}\sqrt{\frac{2}{\pi}}\frac{c}{c^2 + (z-m)^2}, \tag{16}$$

where we have removed a division by $\sqrt{2\pi}$ in the usual statement of the Cauchy. That this is the correct normalization for the amplitude function in Eq.(16) follows from the integral:

$$\int_{-\infty}^{\infty}[A(z]^2 dz = \frac{2}{\pi}\int_{-\infty}^{\infty}\frac{c^3}{[c^2+(z-m)^2]^2}dz = \frac{2}{\pi}\int_{-\infty}^{\infty}\frac{1}{[1+y^2]^2}dy, \tag{17}$$

where we have used the substitution $y = \frac{z-m}{c}$. We may now appeal to the relations, for $a, c > 0$ and n a positive integer:

$$\int\frac{dx}{(ax^2+c)^n} = \frac{1}{2(n-1)c}\frac{x}{(ax^2+c)^{n-1}} + \frac{2n-3}{2(n-1)c}\int\frac{dx}{(ax^2+c)^{n-1}} \tag{18}$$

and

$$\int\frac{dx}{ax^2+c} = \frac{1}{\sqrt{ac}}\tan^{-1}[x\sqrt{\frac{a}{c}}]. \tag{19}$$

Thus we get

$$\frac{2}{\pi}\int_{-\infty}^{\infty}\frac{dy}{[1+y^2]^2} = \frac{2}{\pi}\frac{1}{2}\int_{-\infty}^{\infty}\frac{dy}{1+y^2} = \frac{2}{\pi}\frac{1}{2}2\int_{0}^{\infty}\frac{dy}{1+y^2} = \frac{2}{\pi}\tan^{-1}y]_0^{\infty} = 1. \tag{20}$$

If we further generalize Eq.(1) by relaxing the constraint on β, we obtain the wave function

$$\psi(x,0) = A_o\exp\{[imx - c|x|^{\alpha}][1+i\beta(z/|z|)\tan(\pi\alpha/2)]\}. \tag{21}$$

Note for the wave function in Eq.(21) that since i multiplies β, the normalizing constant A_o given in Eq.(2) is unchanged in terms of α. For $\alpha = \frac{1}{2}$, which we will now consider, $A_o = c = (c')^{\frac{1}{2}}$. Thus for $\alpha = \frac{1}{2}$ and $\beta = -1$, we obtain for the amplitude function the completely positive stable distribution (sometimes called Pearson V), multiplied by the normalizing constant c:

$$A(z) = A_o s_{\frac{1}{2},-1}(z;m,c) = c\frac{c}{\sqrt{(z-m)^3}}\exp[-\frac{c^2}{2(z-m)}]. \tag{22}$$

As a check, we integrate the probability function $P(z) = [A(z)]^2$ corresponding to the amplitude function in Eq.(22):

$$\int_{-\infty}^{\infty}[A(z]^2 dz = \int_{m}^{\infty}\frac{c^4}{(z-m)^3}\exp[-\frac{c^2}{(z-m)}]dz = \int_0^{\infty}\frac{1}{c^2}u^6\frac{2c^2}{u^3}e^{-u^2}du \tag{23}$$

$$= 2\int_0^{\infty}u^3 e^{-u^2} = 2\frac{1}{2}\Gamma(\frac{4}{2}) = 1, \tag{24}$$

where we have used the substitution $u = \frac{c}{(z-m)^{\frac{1}{2}}}$.

Finally, for the general case, we may express the amplitude function as a renormalized stable density, which is in turn represented by a Taylor expansion in the form of gamma functions [2][p. 583] (alternative expansions may be found in [1]):

$$A(z) = A_o s_{\alpha,\beta}(z;0,1) = A_o \frac{1}{z}\sqrt{\frac{2}{\pi}} \sum_{k=1}^{\infty} \frac{\Gamma(1+k/\alpha)}{k!}(-z)^k \, \sin[\frac{k\pi}{2\alpha}(\beta-\alpha)], \tag{25}$$

for $z > 0$ and $1 < \alpha < 2$. For $z < 0$ we have the general relation $s_{\alpha,\beta}(-z;m,c) = s_{\alpha,-\beta}(z;m,c)$. For $0 < \alpha < 1$ we have the similar expansion, for $z > 0$,

$$A(z) = A_o s_{\alpha,\beta}(z;0,1) = A_o \frac{1}{z}\sqrt{\frac{2}{\pi}} \sum_{k=1}^{\infty} \frac{\Gamma(1+k\alpha)}{k!}(-z^{-\alpha})^k \, \sin[\frac{k\pi}{2}(\beta-\alpha)]. \tag{26}$$

We may recover m and c in Eqs.(25,26) by the substitution $z = \frac{u-m}{c^{\frac{1}{\alpha}}}$.

The uncertainty relation

Now let's consider the uncertainty relation. From Eq.(1), where the distribution is symmetric, we get the value for $(\triangle x)^2$ as:

$$(\triangle x)^2 = \int_{-\infty}^{\infty} \psi^*(x,0) x^2 \psi(x,0) dx. \tag{27}$$

Inserting a factor of $u^2 = (2c)^{\frac{2}{\alpha}} x^2$ into the calculation of Eq.(5), we obtain

$$(\triangle x)^2 = \frac{1}{(2c)^{\frac{2}{\alpha}}} \frac{\Gamma(\frac{3}{\alpha})}{\Gamma(\frac{1}{\alpha})}. \tag{28}$$

For $\alpha = 2$ this yields $(\triangle x)^2 = \frac{1}{4c}$, or in Heisenberg's formulation $\frac{1}{4\pi\tau}$.

Next consider the uncertainty in z (or σ). First consider the case $\alpha = 2$. From Eq.(15) we have

$$(\triangle\sigma)^2 = \int_{-\infty}^{\infty} (\sigma-\sigma_o)^2 [A(\sigma)]^2 d\sigma = (\frac{2}{\tau})^{\frac{1}{2}} \int_{-\infty}^{\infty} (\sigma-\sigma_o)^2 \exp[-\frac{2\pi(\sigma-\sigma_o)^2}{\tau}] d\sigma = \frac{\tau}{4\pi}. \tag{29}$$

Thus we obtain the uncertainty relation

$$\triangle x \triangle\sigma = \frac{1}{4\pi}. \tag{30}$$

From the de Broglie relation $\triangle p = h\triangle\sigma$, where h is Planck's constant, this becomes

$$\triangle x \triangle p = \frac{\hbar}{2}. \tag{31}$$

However, for comparison with the results below, we will use for the (renormalized) Gaussian amplitude, the uncertainty relation in the form

$$\triangle x \triangle z = \frac{1}{2}. \tag{32}$$

Note that for the Cauchy density, where $\alpha = 1$, $\beta = 0$, the mean and variance don't exist ("are infinite"). But we are considering a Cauchy *amplitude*, and hence the *square* of the Cauchy density (renormalized) for the probability density. For this density the second moment exists, as we will now demonstrate. From Eqs.(16,17), we calculate $(\triangle z)^2$ as:

$$(\triangle z)^2 = \int_{-\infty}^{\infty} (z-m)^2 [A(z)]^2 dz = \frac{2}{\pi} \int_{-\infty}^{\infty} \frac{c^3 (z-m)^2}{[c^2 + (z-m)^2]^2} dz \tag{33}$$

$$= \frac{2c^2}{\pi} \int_{-\infty}^{\infty} \frac{y^2}{[1+y^2]^2} dy = \frac{2c^2}{\pi} \frac{1}{2} 2 \int_0^{\infty} \frac{dy}{1+y^2} = c^2, \tag{34}$$

where we have used the relation

$$\int \frac{x^2 dx}{(ax^2+c)^n} = -\frac{1}{2(n-1)a} \frac{x}{(ax^2+c)^{n-1}} + \frac{1}{2(n-1)a} \int \frac{dx}{(ax^2+c)^{n-1}}. \tag{35}$$

Thus we obtain the uncertainty relation, from Eqs.(28,34),

$$\triangle x \triangle z = \frac{1}{\sqrt{2}}. \tag{36}$$

For the Pearson V amplitude, we have from Eqs. (22,23)

$$(\triangle z)^2 = \int_{-\infty}^{\infty} (z-m)^2 [A(z)]^2 dz = c^2 \int_m^{\infty} \frac{c^2}{(z-m)} e^{-\frac{c^2}{(z-m)}} dz = c^4 \int_0^{\infty} \frac{1}{y} e^{-y} dy, \tag{37}$$

where we have used the substitution $y = \frac{c^2}{(z-m)}$. This integral is divergent. So instead we calculate

$$\triangle z = \int_{-\infty}^{\infty} |z-m| [A(z)]^2 dz = \int_m^{\infty} \frac{c^4}{(z-m)^2} e^{-\frac{c^2}{(z-m)}} dz = c^2 \int_0^{\infty} e^{-y} dy = c^2. \tag{38}$$

This yields, from Eqs.(28,38) the uncertainty relation

$$\triangle x \triangle z = \sqrt{\frac{15}{2}}. \tag{39}$$

It is easy to see from Eq.(28) that the general uncertainty relation, as a function of α, is

$$\triangle x \triangle z = \sqrt{\frac{1}{(2)^{\frac{2}{\alpha}}} \frac{\Gamma(\frac{3}{\alpha})}{\Gamma(\frac{1}{\alpha})}}. \tag{40}$$

This, then, is the reformulation of Heisenberg's uncertainty relation. The uncertainty is a function of the characteristic exponent α of the (renormalized) stable amplitude. As $\alpha \to 0$, the uncertainty becomes unbounded.

The time-dependent wave function and the dispersion relation

We can write the time-dependent wave equation corresponding to Eq.(1) as a superposition of plane waves:

$$\psi(x,t) = \int_{-\infty}^{\infty} A(z)\ \exp[i(zx - \nu(z)t)]dz, \tag{41}$$

where $A(z)$ is the stable amplitude—a renormalized stable density, and $\nu(z)$ is the frequency. A dispersion relation connects $\nu(z)$ to z.

From the de Broglie relations

$$E = h\nu \tag{42}$$

$$p = hz \tag{43}$$

we obtain the relation

$$\nu = z\frac{E}{p}, \tag{44}$$

which gives as the time-dependent wave equation

$$\psi(x,t) = \int_{-\infty}^{\infty} A(z)\ \exp[iz(x - \frac{E}{p}t)]dz. \tag{45}$$

(Note that we do *not* insert the classical relation $E = \frac{p^2}{2M}$, where M is mass, at this point, because doing so does not yield a proper inverse Fourier transform.) Each plane wave equation $g(x,t) =\ \exp[iz(x - \frac{E}{p}t)]$ has differential operators $\frac{\partial^2}{\partial x^2}$ and $\frac{\partial^2}{\partial t^2}$ with eigenvalues $-z^2$ and $-z^2\frac{E^2}{p^2}$ respectively:

$$\frac{\partial^2 g}{\partial x^2} = -z^2 g \tag{46}$$

$$\frac{\partial^2 g}{\partial t^2} = -z^2\frac{E^2}{p^2}g. \tag{47}$$

These relations give rise to the partial differential equation

$$\frac{\partial^2 g}{\partial t^2} = \frac{E^2}{p^2}\frac{\partial^2 g}{\partial x^2}. \tag{48}$$

The time-dependent wave equation in Eq.(45) may be rewritten more fully (for $\alpha \neq 1$) as

$$\psi(x,t) = A_o\ \exp\{[im(x - \frac{E}{p}t) - c|(x - \frac{E}{p}t)|^\alpha][1 + i\beta((x - \frac{E}{p}t)/|(x - \frac{E}{p}t)|)\tan(\pi\alpha/2)]\}. \tag{49}$$

For symmetric distributions ($\beta = 0$), the probability density function corresponding to $\psi(x,t)$ is

$$P(x,t) = \psi^*(x,t)\psi(x,t) = A_o^2 \exp[-2c|(x - \frac{E}{p}t)|^\alpha], \tag{50}$$

which is the characteristic function of a stable density. For $\alpha = 2$, this is the Heisenberg time-dependent density.

Schrödinger's equation revisted

Schrödinger's equation may be viewed as a simple consequence of the Heisenberg uncertainty relations. Eq.(49) is a solution of the partial differential equation Eq.(48), so we have, as replacement for the Schrödinger equation, the partial differential equation

$$\frac{\partial^2\psi}{\partial t^2} = \frac{E^2}{p^2}\frac{\partial^2\psi}{\partial x^2}, \tag{51}$$

which may be rewritten in the form

$$\nabla^2\psi = \frac{1}{v^2}\frac{\partial^2\psi}{\partial t^2}. \tag{52}$$

This, of course, is the equation of a vibrating string, where $v = \frac{E}{p}$ is the speed of propagation of the waves. It is a true wave equation, by contrast to Schrödinger's heat equation formalism, which relates $\frac{\partial\psi}{\partial t}$ to $\frac{\partial^2\psi}{\partial x^2}$. In fact, noting from Eq.(49), letting β equal zero for simplicity, and letting sgn y denote sgn $(x - \frac{E}{p}t)$, that

$$\frac{\partial\psi}{\partial x} = (im - c\alpha|x - \frac{E}{p}t|^{\alpha-1}\ \mathrm{sgn}\ y)\psi \tag{53}$$

$$\frac{\partial^2\psi}{\partial x^2} = ((im - c\alpha|x - \frac{E}{p}t|^{\alpha-1}\ \mathrm{sgn}\ y)^2 - c\alpha(\alpha-1)|x - \frac{E}{p}t|^{\alpha-2})\psi \tag{54}$$

$$\frac{\partial\psi}{\partial t} = -\frac{E}{p}\frac{\partial\psi}{\partial x} \tag{55}$$

$$\frac{\partial^2\psi}{\partial t^2} = \frac{E^2}{p^2}\frac{\partial^2\psi}{\partial x^2} \tag{56}$$

it does not appear to be particularly useful to relate $\frac{\partial\psi}{\partial t}$ to $\frac{\partial^2\psi}{\partial x^2}$, although this can be done. In fact,

$$\frac{\partial\psi}{\partial t} = -\frac{E}{p}\frac{(im - c\alpha|x - \frac{E}{p}t|^{\alpha-1}\ \mathrm{sgn}\ y)}{((im - c\alpha|x - \frac{E}{p}t|^{\alpha-1}\ \mathrm{sgn}\ y)^2 - c\alpha(\alpha-1)|x - \frac{E}{p}t|^{\alpha-2})}\frac{\partial^2\psi}{\partial x^2}. \tag{57}$$

Only in the case of the Cauchy amplitude $\alpha = 1$ do we find this latter formulation in a simplified form:

$$\frac{\partial \psi}{\partial t} = -\frac{E}{p} \frac{1}{(im - c)} \frac{\partial^2 \psi}{\partial x^2}. \tag{58}$$

If we now make the substitutions $E = \frac{p^2}{2M}$, $p = h\sigma$ we obtain

$$\frac{\partial \psi}{\partial t} = \frac{h\sigma}{2M} \frac{im + c}{(m^2 + c^2)} \frac{\partial^2 \psi}{\partial x^2} \tag{59}$$

which may be rewritten

$$ih\frac{\partial \psi}{\partial t} = -\frac{h^2\sigma}{2M} \frac{m - ic}{(m^2 + c^2)} \frac{\partial^2 \psi}{\partial x^2}. \tag{60}$$

It would appear that the traditional Schrödinger equation involves a hidden Cauchy amplitude assumption. The latter equation can be divided into two equations, one involving m and the other involving $-ic$.

Conclusion

Stable distributions are the only distributions that exist as limit distributions of sums of random variables, thus giving rise to central limit theorems. Therfore they play a paramount role in the physical world. We have shown that Heisenberg's original choice for a wave packet to illustrate his uncertainty principle is simply the characteristic function (the inverse Fourier transform) of a Gaussian distribution, leading to a Gaussian amplitude function with $\alpha = 2$ and $\beta = 0$. Relaxing Heisenberg's assumptions to the general case $0 < \alpha \leq 2$, $|\beta| \leq 1$, leads to stable amplitudes renormalized so that the integral of their squares are probability distributions. The renormalization constant gives rise to a new form of Heisenberg's uncertainty relation, expressed in terms of the characteristic exponent α of the underlying stable amplitude: $\triangle x \triangle z = \sqrt{\frac{1}{(2)^{\frac{2}{\alpha}}} \frac{\Gamma(\frac{3}{\alpha})}{\Gamma(\frac{1}{\alpha})}}$. This relationship was illustrated by explict calculation for the Gaussian ($\alpha = 2$), the Cauchy ($\alpha = 1$), and the Pearson V ($\alpha = \frac{1}{2}$, $\beta = -1$). As $\alpha \to 0$, the uncertainty $\triangle x \triangle z$ becomes unbounded. This means that, depending on the underlying stable amplitude, quantum uncertainty can arise at a macroscopic level.

By eschewing the ad hoc classical insertion $E = \frac{p^2}{2M}$, we were able to solve for the time-dependent wave equation as a superposition of plane waves, by taking the inverse Fourier transform of the stable amplitude function. For $\alpha = 2$, this recovers Heisenberg's case. The

wave function follows the partial differential equation $\frac{\partial\psi}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2\psi}{\partial t^2}$, which is the equation for a vibrating string. This is a proper wave equation, differing from Schrödinger's equation, which is really a heat equation as it relates $\frac{\partial}{\partial t}$, instead of $\frac{\partial^2}{\partial t^2}$, to $\frac{\partial^2}{\partial x^2}$. The traditional form of the Schrödinger equation can be recovered, but only in the case $\alpha = 1$. Thus it would appear that Schrödinger's equation involves a hidden Caucy amplitude assumption. This is not fatal, but is limiting. The more general heat equation relationship is given by Eq.(57).

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* Email: quantum@orlingrabbe.com

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